Europ. J. Combinatorics (1982) **3**, 125–127

An Extremal Problem for two Families of Sets

P. FRANKL

Let r and s be positive integers and $\{A_1A_2, \ldots, A_m\}$, $\{B_1, \ldots, B_m\}$ be two families of sets with $|A_i| = r$, $|B_i| = s$ such that $A_i \cap B_i = \emptyset$ and $A_i \cap B_j \neq \emptyset$ for every $i = 1, \ldots, m$ and $i < j$. We show $m \leqslant \binom{r+s}{s}$. This result generalizes a theorem of Bollobàs. The problem was raised by Pin. The proof uses linear algebra and symmetric tensor products.

1. INTRODUCTION

Let r and s be positive integers and $\mathscr{A} = \{A_1, \ldots, A_m\}$, $\mathscr{B} = \{B_1, \ldots, B_m\}$ be two families of sets such that $|A_i| = r$, $|B_j| = s$ for $1 \leqslant i, j \leqslant m$ and $A_i \cap B_j = \emptyset$ if and only if $i = j$. Bollobàs [1] proved that in this case

$$m \leqslant \binom{r+s}{s}. \tag{1}$$

Katona [3] gave a simpler proof, see also Jaeger and Payan [2].

Bollobàs's result can be formulated in the following way: Suppose the r-uniform hypergraph $\mathscr{A}$ (a family of r-sets) has the property that any $\binom{r+s}{s}$ edges of it can be covered by some s-element set B (i.e., B has non-empty intersection with all these edges), then all the edges of $\mathscr{A}$ can be covered by s elements.

Lovàsz [4] generalized this result to flats of rank r of a matroid, representable over a commutative field.

We obtain the following theorem by modifying Lovàsz's argument.

THEOREM. *Suppose $\mathscr{A} = \{A_1, \ldots, A_m\}$ is a family of r-sets, $\mathscr{B} = \{B_1, \ldots, B_m\}$ is a family of s-sets such that*

(i) $A_i \cap B_i = \emptyset$ *for* $i = 1, \ldots, m$,

(ii) $A_i \cap B_j \neq \emptyset$ *for* $1 \leqslant i < j \leqslant m$.

Then

$$m \leqslant \binom{r+s}{s}. \tag{2}$$

The theorem shows that (1) remains true even if we remove the condition $A_i \cap B_j \neq \emptyset$ for $1 \leqslant j < i \leqslant m$.

This problem was raised by Pin [5], in view of applications to automata theory.

2. PROOF OF THE THEOREM

Let X denote the union of all the A_is and B_js; of course X is a finite set. Thus we may assume that the points of X are in E^{r+1}, the Euclidean space of dimension $r+1$, moreover that these points together with the origin are in general position, i.e., the span of any $r+1$ is a hyperplane of dimension r which does not contain any more points among them. In particular for $i = 1, \ldots, m$ the points of A_i and the origin span a subspace of dimension r, which we denote by V_i, such that $V_i \cap (X - A_i) = \emptyset$. Let us denote further by $\mathbf{u}_i$ its normal unit vector, i.e. $\mathbf{u}_i$ is orthogonal to V_i and it points to the positive side of V_i.

0195–6698/82/020125 + 03 \$02.00/0

For simplicity we shall not distinguish between a point in E^{r+1} and its vector, i.e., the vector joining the origin and this point. Thus $a \in V_i$ is equivalent to $(a, u_i) = 0$.

Let us denote by f_a the linear functional defined by $f_a(v) = (a, v)$. The space of linear functionals is isomorphic to E^{r+1}; let $W_1, \ldots, W_s$ be s disjoint copies of it.

The tensor product of $W_1, \ldots, W_s$ is the vector space of multilinear functions $f(v_1, \ldots, v_s)$, $v_i \in E^{r+1}$ (multilinear means linear in each of its variables). This vector space has dimension $(r+1)^s$ and it is generated by the functions of the form

$$f(v_1, \ldots, v_s) = \prod_{1 \le i \le s} f_i(v_i), \quad \text{where } f_i \in W_i.$$

A multilinear function is called symmetric if its value is invariant under permutations of its variables, i.e. if $\rho(i)$, $i = 1, \ldots, s$ is an arbitrary permutation of $\{1, \ldots, s\}$, then

$$f(v_1, \ldots, v_s) = f(v_{\rho(1)}, \ldots, v_{\rho(s)}).$$

The set of all symmetric multilinear functions is a subspace of dimension $\binom{r+s}{s}$ of the tensor product.

For each of the sets $B_j = \{b_1^j, \ldots, b_s^j\}$ we define a symmetric multilinear function

$$f_{B_j} = f_{B_j}(v_1, \ldots, v_s) = \frac{1}{s!} \sum_{\rho \in S_s} \prod_{1 \le i \le s} f_{b^j_{\rho(i)}}(v_i). \tag{3}$$

These are m multilinear functions in a space of dimension $\binom{r+s}{s}$, thus (2) will follow as soon as we show that these functions are linearly independent.

Suppose the contrary, i.e. there exist real numbers c_j, $j = 1, \ldots, m$, not all zero, such that $f = \sum_j c_j f_{B_j}$ is the zero functional.

Let t be the first j such that $c_j \neq 0$. We want to evaluate $f(u_t, \ldots, u_t)$. From (3) we deduce

$$f_{B_j}(u_t, \ldots, u_t) = \prod_{1 \le i \le s} (b_i^j, u_t). \tag{4}$$

As for $t < j \le m$ we have $B_j \cap A_t \neq \emptyset$, then by definition of u_t the scalar product of u_t and the corresponding b_i^j is zero, thus the value of (4) is zero. As $c_j = 0$ for $1 \le j < t$, we showed

$$c_j f_{B_j}(u_t, \ldots, u_t) = 0 \quad \text{for } j \neq t. \tag{5}$$

On the other hand $B_t \cap A_t = \emptyset$ and the points are in general position. Thus $B_t \cap V_t = \emptyset$ which means that for every $b_i^j \in B_j$ we have $(b_i^j, u_t) \neq 0$, implying

$$c_t f_{B_t}(u_t, \ldots, u_t) \neq 0. \tag{6}$$

Summing up (5) and (6) we obtain the desired contradiction

$$f(u_t, \ldots, u_t) = \sum_j c_j f_{B_j}(u_t, \ldots, u_t) \neq 0.$$

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Received 4 January 1982

P. Frankl
Centre National de la Recherche Scientifique,
54 Boulevard Raspail, 75006 Paris, France